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A FINITENESS THEOREM FOR RICCI CURVATURE IN DIMENSION THREE

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1. Introduction

The purpose of this paper is to prove the following result.

Theorem 1. *There are only finitely many homotopy types in the class of three-dimensional Riemannian manifolds M satisfying*

$$\mathrm{Ric}(M) \geq -H^2, \quad \mathrm{Diam}(M) \leq D, \quad \mathrm{Vol}(M) \geq V,$$

where $\mathrm{Ric}(M)$ *is the Ricci curvature,* $\mathrm{Diam}(M)$, *the diameter, and* $\mathrm{Vol}(M)$, *the volume of* M.

As a noncompact counterpart to Theorem 1, we also prove

Theorem 2. *Let M^3 be a complete open three-manifold satisfying*

$$\mathrm{Ric} \geq 0, \qquad \mathrm{Vol}(B_p(r)) \geq cr^3.$$

Then M is contractible.

Results of the type of Theorem 1, known as finiteness theorems, were first obtained by A. Weinstein [20] and J. Cheeger [4], [12]. Cheeger's finiteness theorem states that there are only finitely many diffeomorphism types for the class of Riemannian manifolds with a bound on the absolute value of sectional curvature and bounds on the diameter and volume identical to that of Theorem 1. Subsequently, Grove-Petersen [7] proved the finite homotopy type theorem only assuming a lower bound on sectional curvature. This was later strengthened to finite diffeomorphism types for $n \neq 3, 4$ by Grove-Petersen-Wu [8]. Theorem 1 is an attempt to generalize this to an assumption on Ricci curvature instead of sectional curvature.

All finiteness theorems as quoted above prove that, under a bound on the curvature (which is local), the topology of the manifold is controlled by its size. The proofs of these theorems rely on the understanding of the local structure of the corresponding class. For Cheeger's finiteness theorem, the crucial step is to prove a lower bound on the injectivity radius for that class, hence on a uniform size smaller than that bound, the topology is simple: it is diffeomorphic to a Euclidean ball. The corresponding statement for the

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result of Grove-Petersen is a lower bound on the *geometric contractibility radius* (for definition see §3), which says a small ball is contractible, relative to a bigger ball. Theorem 1 is proved by establishing a similar bound on the geometric contractibility radius. Let us point out that the existence of such a bound is strictly three dimensional. In fact, M. Anderson showed there are metrics on $S^2 \times S^2$ satisfying the same bound as in Theorem 1, but with no bound on the geometric contractibility radius [1]. This is one of the main difficulties in trying to generalize Theorem 1 to higher dimensions.

Then what is special about dimension three? The first thing to come into mind is that in dimension-three sectional curvature can be recovered from Ricci curvature (an alternating sum). This has been used by R. Hamilton in conjunction with Ricci flow to get the spectacular result that any three-manifold with positive Ricci curvature admits a metric with constant sectional curvature. Another theorem special to dimension three is that of Schoen-Yau, which states that a three-dimensional open manifold with positive Ricci curvature is diffeomorphic to R^3. These two results exploit the analytic and geometric aspects of three-dimensional geometry of Ricci curvature. In contrast, our proof exploits the topological aspect. In fact, the geometric part of the argument, such as that contained in Lemma 3.1 to Lemma 3.4, holds for all dimensions. It is the topological part which is special to dimension three. Thus it is perhaps fair to say that, as far as the geometry relevant to finiteness theorems is concerned, we know no more in dimension three than in higher dimensions. An outstanding question is whether we can bound the length of shortest closed geodesics by the data in Theorem 1. This is known not to be true if the dimension is greater than three [1].

The class of manifolds in Theorem 2 can be obtained by scaling the class in Theorem 1. Thus Theorem 2 can be thought of as a local version of Theorem 1. Its validity demonstrates the possibility of bounding the geometric contractibility radius in the proof of Theorem 1. However, Theorem 2 can also be considered as a partial result towards the classification of complete three-manifolds of nonnegative Ricci curvature, originally put forward by Schoen-Yau in [14]. To this date, this classification is still open. However, let us point out that with an additional assumption on the boundedness of sectional curvature, W. Shi [16] and Anderson-Rodriguez [2] did gave a classification. The method of W. Shi is analytic, using Hamilton's Ricci flow for open manifolds as developed in [17], [18], while the method of Anderson-Rodriguez is geometric, through the study of minimal surfaces along the lines of Schoen-Yau [14].

The idea of the proof of our results was very much inspired by the papers of Schoen-Yau [14] and M. Anderson [1]. The results of this paper have been announced under the same title in [22].

2. Proof of the open case

In this section we give the proof of Theorem 2. Our argument follows closely that of Schoen-Yau [14]. The strategy is to prove that $\pi_1(M) = \pi_2(M) = 0$. Since m is open and of dimension three, $H_k(M, Z) = 0$, for all $k \geq 3$. By the Hurewicz Theorem, we have $\pi_k(M) = 0$ for all $k \geq 1$. Hence M is contractible by the Whitehead Theorem.

Let us first prove $\pi_2(M) = 0$. If $\pi_2(M) \neq 0$, then $\pi_2(\widetilde{M}) \neq 0$, where $\widetilde{M}$ is the universal covering space of M. The Sphere Theorem in three-dimensional topology [10] says that there exists an embedded S^2 in $\widetilde{M}$ which is not homotopically trivial. If $\widetilde{M}\backslash S^2$ were connected, we could take a loop in $\widetilde{M}$ intersecting S^2 at exactly one point. This loop could not be null-homotopic. This would contradict $\pi_1(\widetilde{M}) = \{e\}$. Thus S^2 divides $\widetilde{M}$ into two connected components. By Van Kampen's theorem, each component is simply connected. If one of these were compact, then by the Hurewicz Theorem it is trivial in π_2 since S^2 is a trivial element in H_2 of the compact set. This is a contradiction. Therefore, S^2 divides $\widetilde{M}$ into two noncompact components. This implies the existence of a line, namely a geodesic which is minimizing between any two of its points. Now the Cheeger-Gromoll Splitting Theorem [5] implies that $\widetilde{M}$ is a product of a line and a compact two-manifold Σ. Let $T_r(\Sigma)$ be the r tubular neighbourhood of Σ of radius r. Then $\mathrm{Vol}(T_r(\Sigma)) = r \cdot \mathrm{Vol}(\Sigma)$. It is easy to see that the volume growth condition in Theorem 1.2 is satisfied for any point. We can thus assume $p \in \Sigma$. Then it follows that $\mathrm{Vol}(B_p(r)) \leq \mathrm{Vol}(T_{r+\mathrm{Diam}(\Sigma)}(\Sigma)) = \mathrm{Vol}(\Sigma) \cdot (r + \mathrm{Diam}(\Sigma)) \leq r^2$, for r large enough. This contradicts our assumption on the volume growth. Hence $\pi_2(M) = 0$.

Since $\dim(M) = 3$ and M is open, thus $H_k(M, Z) = 0$ for $k \geq 3$. By the Hurewicz Theorem, all higher homotopy groups of M vanish. Therefore M is a $K(\pi, 1)$ space, and $H^i(\pi_1(M)) = H^i(M) = 0$, for $i \geq 3$. Since infinitely many cohomology groups of a finite cyclic group are nonzero, hence $\pi_1(M)$ is torsion free.

We now prove that $\pi_1(M)$ is trivial. By passing to a covering space of M, we may assume $\pi_1(M) = Z$. By using a volume comparison argument similar to the one in [1] we will show this is impossible. Fix a point $p \in M$

and $\tilde{p} \in \widetilde{M}$, such that $\pi(\tilde{p}) = p$, where π is the covering map. Let σ be a geodesic loop at p representing a generator for $\pi_1(M, p)$, and F be a fundamental domain of M containing $\tilde{p}$. Then it is obvious that

$$\bigcup_{k=1}^{N} [\sigma]^k (F \cap B_{\tilde{p}}^{\widetilde{M}}(r)) \subset B_{\tilde{p}}^{\widetilde{M}}(N \cdot L(\sigma) + r),$$

where $L(\sigma)$ is the length of σ. Notice that π is volume-preserving when restricted to F. Then using $\mathrm{Vol}([\sigma](F) \cap F) = 0$, we obtain

$$\begin{aligned} N \cdot \mathrm{Vol}(B_p(r)) &= N \cdot \mathrm{Vol}(F \cap B_{\tilde{p}}^{\widetilde{M}}(r)) \\ &= \mathrm{Vol}\left(\bigcup_{k=1}^{N} [\sigma]^k (F \cap B_{\tilde{p}}^{\widetilde{M}}(r))\right) \\ &\leq \mathrm{Vol}(B_{\tilde{p}}^{\widetilde{M}}(N \cdot L(\sigma) + r)) \\ &\leq \frac{4}{3}\pi (N \cdot L(\sigma) + r)^3 \quad (\text{since } \mathrm{Ric} \geq 0). \end{aligned}$$

Choosing $N \geq [\frac{32\pi}{6c}]$, and $r \geq N \cdot L(\sigma)$, we have

$$\mathrm{Vol}(B_p(r)) \leq \frac{4\pi}{3N} \cdot (2r)^3 \leq \frac{c}{2} r^3.$$

This is a contradiction. Thus, $\pi_1(M) = \{e\}$. Therefore all homotopy groups of M vanish. We hence conclude that M is contractible by the Whitehead Theorem.

3. Proof of the finiteness theorem

Let us denote by $\mathscr{M}(n)$ the class of n-dimensional manifolds satisfying the bounds: $\mathrm{Ric} \geq -(n-1)H$, $\mathrm{Diam} \leq D$, $\mathrm{Vol} \geq V$. As pointed out in the introduction, the crucial step towards a finiteness theorem is to get a control of the local topology. For the class $\mathscr{M}(3)$, this takes the form of a lower bound on the geometric contractibility radius. By the examples of Sha-Yang and Anderson [1], such a bound does not exist for $\mathscr{M}(n)$ when $n > 3$.

We first define the geometric contractibility radius (of relative size R):

$$C_R(M) = \inf_{p \in M} \sup\{r | B_p(r) \text{ is contractible in } B_p(R \cdot r)\}.$$

The crucial step in proving Theorem 1 is the following proposition.

Proposition 3.1. *There exist constants* R, r_0 *depending only on* H, D, V, *such that*

$$C_R(M) \geq r_0$$

for any $M \in \mathscr{M}(3)$.

By the Gromov precompactness theorem, $\mathscr{M}(n)$ is precompact with respect to the Hausdorff distance [6]. It is of great interest to know the structure of limiting spaces in this class. Appealing to a result of Grove-Petersen-Wu [8], Proposition 3.1 immediately implies the following.

Corollary 3.1. *If* $M_i \in \mathscr{M}(3)$, *and* $X = \lim M_i$, *where the limit is taken with respect to the Hausdorff distance, then* X *is a homology manifold.*

We devote the rest of this section in proving Proposition 3.1. We begin the proof with a few lemmas. These lemmas hold for all dimensions. The restriction to dimension three is only needed at the end of the proof.

The following lemma is well known. We present it here in order to fix some constants.

Lemma 3.1. *There exist constants* C_1, C_2 *and* d *depending only on* n, H, D, V, *such that, for any* $M \in \mathscr{M}(n)$, $p \in M$, *we have,*

$$C_1 r^n \leq \mathrm{Vol}(B_p(r)) \leq C_2 r^n, \qquad 0 \leq r \leq D,$$

and

$$\mathrm{Diam}_p(M) \geq d,$$

where $\mathrm{Diam}_p(M) = \sup\{d(p, q) | q \in M\}$.

Proof. By the Bishop volume comparison theorem,

$$\begin{aligned} \mathrm{Vol}(B_p(r)) &\leq \mathrm{Vol}^H(B(r)) \\ &= \int_0^r \left(\frac{\sinh \sqrt{H} t}{\sqrt{H}}\right)^{n-1} dt \\ &\leq C_2 r^n, \qquad 0 \leq r \leq D, \end{aligned}$$

where

$$C_2 = \sup_{0 \leq r \leq D} \frac{1}{r^n} \int_0^r \left(\frac{\sinh \sqrt{H} t}{\sqrt{H}}\right)^{n-1} dt.$$

Similarly, by the Bishop-Gromov relative volume comparison theorem, we obtain

$$\begin{aligned}\mathrm{Vol}(B_p(r)) &\geq \frac{\mathrm{Vol}^H(B(r))}{\mathrm{Vol}^H(B(D))}\cdot \mathrm{Vol}(B_p(D))\\ &\geq \frac{V}{\mathrm{Vol}^H(B(D))}\cdot \int_0^r \left(\frac{\sinh\sqrt{H}t}{\sqrt{H}}\right)^{n-1} dt\\ &\geq C_1 r^n,\end{aligned}$$

where

$$C_1 = \frac{V}{\mathrm{Vol}^H(B(D))}\cdot \inf_{0\leq r\leq D}\frac{1}{r^n}\int_0^r \left(\frac{\sinh\sqrt{H}t}{\sqrt{H}}\right)^{n-1} dt.$$

For the diameter, since

$$V \leq \mathrm{Vol}(M) \leq C_2(\mathrm{Diam}_p(M))^n,$$

we have

$$\mathrm{Diam}_p(M) \geq \left(\frac{V}{C_2}\right)^{1/n}.$$

Lemma 3.2. *There exist constants $R_1(n, H, D, V)$, $r_1(n, H, D, V)$ such that for any $M^n \in \mathscr{M}(n, H, D, V)$, $p \in M$, and $s \leq r_1$, $B_p(R_1\cdot s)\backslash B_p(s)$ has at most one component whose intersection with $\partial B_p(R_1\cdot s/3)$ is nonempty.*

Proof. We prove this by contradiction. Let C_1 and C_2 be two such components. Without loss of generality we can assume that

$$\mathrm{Vol}(C_1\cap B_p(R\cdot s/3)) \leq \mathrm{Vol}(C_2\cap B_p(R\cdot s/3)).$$

Thus,

$$\mathrm{Vol}(B_p(R\cdot s/3)\backslash B_p(s)) \leq 2\mathrm{Vol}(B_p(R\cdot s/3)\backslash(C_1\cap B_p(R\cdot s/3))).$$

Take $Q_1 \in C_1\cap \partial B_p(R\cdot s/3)$. Since every minimal geodesic γ with $\gamma(l)\in B_p(R\cdot s/3)\backslash B_p(s)$ satisfies $l\leq \frac{2}{3}R\cdot s$, and $\gamma(t)\in B_p(s)$ for some t satisfying $\frac{1}{3}R\cdot s - s\leq t\leq \frac{1}{3}R\cdot s + s$, we have

$$B_p(\tfrac{1}{3}R\cdot s)\backslash(C_1\cap B_p(\tfrac{1}{3}R\cdot s)) \subset T_{\frac{1}{3}R\cdot s-s,\,(2/3)R\cdot s}(Q_1),$$

where T_{r_1,r_2} is the annulus of radius r_1 and r_2. Thus the triangle inequality implies that

$$T_{(1/3)R\cdot s-s,\,(1/3)R\cdot s+s}(Q_1)\subset B_p(3s),$$

so that

$$\begin{aligned}\frac{\mathrm{Vol}(B_p(R\cdot s/3)\backslash B_p(s))}{\mathrm{Vol}(B_p(3s))} &\le 2\frac{\mathrm{Vol}(B_p(R\cdot s/3)\backslash(C_1\cap B_p(R\cdot s/3)))}{\mathrm{Vol}(B_p(3s))}\\ &\le 2\frac{\mathrm{Vol}(T_{(1/3)R\cdot s-s,\,(2/3)R\cdot s}(Q_1))}{\mathrm{Vol}(T_{(1/3)R\cdot s-s,\,(1/3)R\cdot s+s}(Q_1))}\\ &\le 2\frac{\mathrm{Vol}^H_{(1/3)R\cdot s-s,\,(2/3)R\cdot s}}{\mathrm{Vol}^H_{(1/3)R\cdot s-s,\,(1/3)R\cdot s+s}}\\ &\le C_3(n,H,D)\cdot R,\end{aligned}$$

where we have denoted by $\mathrm{Vol}^H_{r_1,r_2}$ the volume of an annulus of radius r_1 and r_2 in the space of constant curvature $-(n-1)H$. Together with Lemma 2.2, the above yields

$$\frac{C_1\cdot(R/3)^n - C_2}{C_2\cdot 3^n} \le C_3\cdot R.$$

This is impossible if we choose $R(n,H,D,V)$ big enough. In the proof, we also need that $s\cdot R\le d$. Thus $s\le d/R_1 = r_1$.

Lemma 3.3. *There are constants R_2, r_2 and N depending only on n, H, D, V, such that for any $M\in\mathscr{M}(n)$, $p\in M$, $r\le r_2$, if $I\colon B_p(r)\to B_p(R\cdot r)$, then any subgroup of G of $I_*(\pi_1(B_p(r)))$ satisfies*

$$\mathrm{order}(G)\le N.$$

In particular, there is no element of infinite order in $I_*(\pi_1(B_p(r)))$ whenever $r\le r_2$.

Proof. This is basically the same as in [1] or as in the proof of Theorem 2. But let us point out that we are working with a metric ball, which is not complete. Hence its universal covering space with the pulled back metric is also not complete. Since we need to use the Bishop volume estimate for geodesic balls, we have to show it is still valid in this case. This turns out to be fairly easy in our situation since we are working with a relative version. Namely, although $\widetilde{B_p(r)}$ is not complete, in the universal covering space of a larger ball $B_p(R_2\cdot r)$, $B_{\tilde p}(Nr)$ $(N\ll R_2)$ is a usual metric ball, and hence Bishop's volume estimate still holds. We will address the problem in the proof.

It is a well-known fact that we can choose a set of generators $\{[\sigma_i]\}$ for $I_*(\pi_1(B_p(r)))$, such that $\mathrm{length}(\sigma_i)\le 2r$ and there is a bound on the number of generators [19], say $k(n,D,V,H)$. Let V be the universal covering space of $B_p(R\cdot r)$ with the pulled back metric. Pick $\tilde p\in V$, such that $\pi(\tilde{p}) = p$. Let F be a fundamental domain of the covering with $\tilde{p} \in F$. Denote $U(m) = \{\text{element of } G \text{ of word length} \leq m\}$. It is easy to see that $\#U(m) \geq m$ unless $U(m) = G$. (This is because $\#U(m+1) > \#U(m)$ unless $U(m) = G$). Let $m_0 = [(\frac{3}{2})^n \cdot C_2/C_1] + 1$. Consider,

$$B = \bigcup_{g \in U(m_0)} g(F \cap \pi^{-1}(B_p(2m_0 r))).$$

Take any point $x \in B$, and a curve γ from x to $\tilde{p}$. Then $\pi(\gamma)$ is a curve from $\pi(x) \in B_p(2m_0 r)$ to p. Let σ be a minimal geodesic in the homotopy class of $\pi(\gamma)$ keeping the end points fixed. Then $\text{length}(\sigma) \leq m_0 \cdot \sup_i\{\text{length}(\sigma_i\} + m_0 \cdot r \leq 3m_0 r$. If we choose $R_2 = 6m_0$, then σ is a smooth geodesic. Lifting σ to V, we get a smooth geodesic from $\tilde{p}$ to x. What we have proved is that any point in B can be joined to $\tilde{p}$ by a smooth geodesic of length $\leq 3m_0 r$. It thus follows from the proof of Bishop volume comparison theorem that

$$\text{Vol}(B) \leq \text{Vol}^H(3m_0 r).$$

If $\text{order}(G) > \#U(m_0)$, then

$$\begin{aligned} m_0 \text{Vol}(B_p(2m_0 r)) &\leq \#U(m_0)\text{Vol}(F \cap \pi^{-1}(B_p(r))) \\ &= \text{Vol}(B) \leq \text{Vol}^H(3m_0 r). \end{aligned}$$

Therefore,

$$m_0 \leq \frac{\text{Vol}^H(3m_0 r)}{\text{Vol}(B_p(2m_0 r))}.$$

Let $r_2 = D/(6m_0)$. Then for any $r \leq r_2$, we have $3m_0 r < D$. It thus follows from Lemma 3.2 that

$$m_0 \leq \frac{C_2(3m_0 r)^n}{C_1(2m_0 r)^n} = \left(\frac{3}{2}\right)^n \cdot \frac{C_2}{C_1}.$$

This contradicts the choice of m_0. Hence $\text{order}(G) \leq \#U(m_0) \leq k^{m_0} = N$.

Lemma 3.4. *Let K be a compact Riemannian manifold, and $\widetilde{K}$ a k-fold covering of K with the pulled back metric. Then*

$$\text{Diam}(\widetilde{K}) \leq 2k \cdot \text{Diam}(K).$$

Proof. We denote by Γ the group of deck transformations, $\#\Gamma = k$. Fix a point $p \in K$, and $\tilde{p} \in \widetilde{K}$ such that $\pi(\tilde{p}) = p$. Let F be the Dirichlet fundamental domain of the covering, that is, let

$$F = \{x \in \widetilde{K} | d(x, \tilde{p}) \leq d(\gamma x, \tilde{p}), \text{ for any } \gamma \in \Gamma\}.$$

We first show that for any $x \in F$, $d(x, \tilde{p}) \leq \mathrm{Diam}(K)$. Indeed, let σ be a minimal geodesic from $\tilde{p}$ to x, with $\sigma(l) = x$. Then $\pi \circ \sigma$ is a curve from p to $\pi(x)$ with $\mathrm{length}(\pi \circ \sigma) = \mathrm{length}(\sigma) = l$. If $l > \mathrm{Diam}(K)$, there exists a curve α from p to $\pi(x)$ with length $l_1 \leq \mathrm{Diam}(K) < l$. Lift α to $\widetilde{K}$ with $\alpha(0) = \tilde{p}$. Then $d(\alpha(l_1), \tilde{p}) \leq l_1 < l$. But $\pi(\alpha(l_1)) = \pi(x)$, so $\alpha(l_1) = \gamma x$ for some $\gamma \in \Gamma$. This contradicts the definition of F. Hence $d(x, \tilde{p}) \leq \mathrm{Diam}(K)$.

Now for any two points x and y in $\widetilde{K}$, let γ be a curve connecting them. $\widetilde{K}$ is the union of k Dirichlet fundamental domains with centers at $\pi^{-1}(p)$. Since ∂F has measure zero, we can choose the curve γ such that it has the property that $\gamma \cap \partial F$ has no accumulation points. Thus $\gamma \cap \partial F$ is a finite set. Therefore for each fundamental domain F, we can pick the point where γ first enters F and the point where γ last leaves F, say at $\gamma(t_1)$ and $\gamma(t_2)$. We can replace the segment $\gamma([t_1, t_2])$ by a curve from $\gamma(t_1)$ to $\tilde{p}$ and then from $\tilde{p}$ to $\gamma(t_2)$. The previous paragraph shows that we can choose this curve to have length $\leq 2\mathrm{Diam}(K)$, Continuing this process we get a curve from x to y which intersects each fundamental domain only once and inside each fundamental domain it has length at most $2\mathrm{Diam}(K)$. Thus, $d(x, y) \leq 2k \cdot \mathrm{Diam}(K)$. Therefore $\mathrm{Diam}(\widetilde{K}) \leq 2k \cdot \mathrm{Diam}(K)$.

Remark. In the statement of Lemma 3.4, we assumed that K is a Riemannian manifold. From the proof we see that the same statement holds for a much larger class of objects. In particular, it holds for compact (smooth) metric balls $B_p(r) \subset (M, g)$. In the proof of Proposition 3.1, we will use Lemma 3.4 in this form.

Next we prove a topological lemma concerning fundamental groups of three-dimensional manifolds. This will be needed in the proof of Proposition 3.1 for the orientable case. A basic reference on this subject is the book by J. Hempel [10].

Lemma 3.5. *Let $M \subset \mathrm{int}(N)$ be two compact orientable three-manifolds with nonempty boundary. If $\pi_2(M) \to \pi_2(N)$ is trivial, then $\pi_1(M)$ is torsion free.*

Proof. Let $M = \#_{i=1}^{k} M_i$ be a prime decomposition of M. Since $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k)$, a free product, we can assume that M itself is prime. Without loss of generality, we can assume $\pi_1(M) \neq \{e\}$. We first prove that $\pi_1(M)$ is infinite. In fact, if we denote by $\widehat{M}$ the manifold by capping off all two-spheres in ∂M by three balls, then we claim that $\widehat{M}$ is not closed. If it were, then ∂M would only consist of two-spheres. Since $\pi_2(M) \to \pi_2(N)$ is trivial, each S^2 in ∂M

separates N. Moreover, at least one component of $N-S^2$ is compact and simply connected. In fact, if neither components were compact, it would follow from Poincaré duality that each such S^2 is nontrivial in $\pi_2(N)$. If none of the compact components were simply connected, lifting to the universal covering space, duality would again imply that each such S^2 is nontrivial in $\pi_2(N)$. Hence we have that each S^2 in ∂M bounds a homotopy three-ball in N. If one of such S^2 bounds a three-ball containing M, then M is simply connected. This is a contradiction. So all 2-spheres in ∂M bound in the exterior. By adding these homotopy three-balls to M, we get a closed three-manifold embedded in a three-manifold with nonempty boundary. This is impossible. So $\widehat{M}$ is not closed. By taking the double of $\widehat{M}$, we obtain a closed three-manifold $\widehat{M} \cup_{\partial \widehat{M}} \widehat{M}$. Hence, $0 = \chi(\widehat{M} \cup_{\partial \widehat{M}} \widehat{M}) = \chi(\widehat{M}) + \chi(\widehat{M}) - \chi(\partial \widehat{M})$. So $\chi(\widehat{M}) = \frac{1}{2}\chi(\partial \widehat{M}) \leq 0$. On the other hand, $\chi(\widehat{M}) = 1 - b_1(\widehat{M}) + b_2(\widehat{M}) - b_3(\widehat{M})$, where b_i is the ith Betti number. Since $\widehat{M}$ is not closed, we have $b_3(\widehat{M}) = 0$. Therefore $b_1(\widehat{M}) \geq 1 + b_2(\widehat{M}) \geq 1$. By Mayer-Vitoris sequence, we obtain $b_1(M) = b_1(\widehat{M}) \geq 1$. So $\pi_1(M)$ is infinite.

By the method of contradiction, we now prove $\pi_1(M)$ is torsion free. If not, let G be a finite subgroup of $\pi_1(M)$, and let m_1 be a covering space of M such that $p_*(\pi_1(M_1)) = G$. Using the same notation as before, we have $\pi_1(\widehat{M}_1) = G$. Let $\widetilde{M}_1$ be the universal covering space of $\widehat{M}_1$. Then $\pi_2(\widetilde{M}_1) = 0$ since $\widehat{M}_1$ is prime and orientable. We claim $\widetilde{M}_1$ is closed. Otherwise, by the Hurewicz theorem, $H_i(\widehat{M}_1) = H_i(\widetilde{M}_1) = 0$ for $i \geq 2$. Hence $H_i(G) = 0$ for $i \geq 2$. This is not possible since G is finite. Therefore $\widehat{M}_1$ is closed. Hence M_1 is compact and its boundary consists of two-spheres. It then follows that $M_1 \xrightarrow{p} M$ is a finite covering. So $\pi_1(M)$ is finite. This is a contradiction. Thus $\pi_1(M)$ is torsion free.

Proof of Proposition 3.1. For any $M \in \mathscr{M}(3)$, $p \in M$, consider the inclusion $I: B_p(r) \to B_p(R \cdot r)$. The precise value of R will be determined in the proof. Just as in the proof of Theorem 2, we first show that I induces trivial maps on π_2 and π_1.

For this part, we have to distinguish between the orientable case and the nonorientable case. The arguments are along a similar line with some difference in details. Let us briefly summarize it here. What we will actually show is that either $\pi_2(B_p(r)) = 0$ or a nontrivial element in $\pi_2(B_p(r))$, which is represented by an embedded S^2 or RP^2 according to orientability, divides $B_p(R \cdot r)$ into two parts; one part is compact and simply connected. Thus, for the orientable case, this implies that I is trivial on π_2.

For the nonorientable case, this implies that $\pi_2(B_p(r)) = 0$. Either case implies $\pi_1(B_p(r))$ is torsion free (the orientable case follows from Lemma 3.5 and the nonorientable case follows form group homology). Then the conclusion that I is trivial on π_1 follows immediately from Lemma 3.3.

In what follows, we first treat the orientable case.

If I is not trivial on π_2, by the sphere theorem in three-dimensional topology, there is an (smoothly) embedded S^2 in $B_p(r)$, representing a nontrivial homotopy class in $B_p(R \cdot r)$. There are three possibilities we have to consider.

Case 1. S^2 does not separate $B_p(R \cdot r)$. From standard three-dimensional topology (Lemma 3.8 in [10]), we have the decomposition $B_p(R \cdot r) = V_1 \# V_2$, where V_1 is a two-sphere bundle over S^1. Hence there is an element $[\sigma] \in \pi_1(V_1)$ of infinity order, and σ is contained in $B_p(R \cdot r)$. Since $B_p(R^2 \cdot r) = V_1 \# V_3$ for some manifold V_3, σ is also an element of infinite order in $B_p(R^2 \cdot r)$. This is impossible by Lemma 3.3. For this case we require that $R^2 \cdot r \leq d/2$, $R \geq R_2$, $R \cdot r \leq r_2$, where r_2, R_2, d are the constants in Lemma 3.1 and 3.3.

Case 2. S^2 separates $B_p(R \cdot r)$ into two components, both of which have nontrivial intersection with $\partial B_p(R \cdot r)$; this is impossible by Lemma 3.2. For this to work we require that $R \geq 3R_1$, $r \leq r_1$, where R_1, r_1 are the constants in Lemma 3.2.

Case 3. S^2 separates $B_p(R \cdot r)$ into two connected components, one of which, M_1, has nontrivial intersection with $\partial B_p(R \cdot r)$, and the other, M_2, is compact with $\partial(M_2) = S^2$. Hence $B_p(R \cdot r) = M_1 \# M_2$. Let us note that M_2 cannot be simply connected. Otherwise the S^2 would be contractible, contradicting our assumption. Hence $\pi_1(M_2)$ is nontrivial and, because of the connected sum decomposition, the inclusion into $\pi_1(B_p(R \cdot r))$ is injective. Since the larger ball $B_p(R^2 \cdot r)$ is also a connected sum of M_2 and another manifold, we conclude that $\pi_1(M_2)$ is also injectively included in $\pi_1(B_p(R^2 \cdot r))$. Notice that $M_2 \subset B_p(R \cdot r)$ (this is why we have to consider the larger ball $B_p(R^2 \cdot r)$). By Lemma 3.3, the order of $\pi_1(M_2)$ is bounded by N. Consider the covering space K of $B_p(R^2 \cdot r)$ as follows. First take the universal covering space $\widetilde{M}_2$ of M_2; then glue $B_p(R^2 \cdot r) \backslash M_2$ to each lifting of S^2, and denote the resulting space as K. Thus the deck transformation group of this covering is $\pi_1(M_2)$. It is obvious from this description that $\widetilde{M}_2$ separates K into $\# \pi_1(M_2)$ components. Now by Lemma 3.4 (see the remark after it), $\text{Diam}(T) \le 2N \cdot \text{Diam}(M_1) \le 2N \cdot 2Rr$. This is again impossible according to Lemma 3.2. For this part we need $R \ge 4N$, $R \ge R_2$, $R \cdot r \le r_2$ and $R^2 \cdot r \le d/2$.

Thus, we have proved that if $I: B_p(r) \to B_p(R \cdot r)$, then I_* is trivial on π_2 whenever $R \ge \max\{3R_1, R_2, 4N\}$ and $r \le \min\{r_1, r_2/R, d/2R^2\}$.

We now show that I_* is trivial on π_1. This is now very easy. In fact, consider the inclusions $B_p(r) \subset B_p(R \cdot r) \subset B_p(R^2 \cdot r)$. From the previous paragraph, if we choose r smaller, say $r \le \min\{r_1/R, r_2/R^2, d/2R^3\}$, then the second inclusion $B_p(R \cdot r) \subset B_p(R^2 \cdot r)$ satisfies the above condition, and hence this inclusion induces a trivial map on π_2. It now follows from Lemma 3.5 that $\pi_1(B_p(R \cdot r))$ is torsion free. Thus, if $I_*: B_p(r) \to B_p(R \cdot r)$ were not trivial on π_1, there would be an element of $\pi_1(B_p(r))$ which is nontrivial in $\pi_1(B_p(R \cdot r))$, hence is necessarily of infinite order in $\pi_1(B_p(R \cdot r))$ since the later is torsion free. This is impossible by Lemma 3.3. Therefore I_* is trivial on π_1.

We have thus proved that for the orientable case, if

$$R \ge \max\{3R_1, R_2, 4N\}, \qquad r \le \min\left\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\right\},$$

then $I_*: B_p(r) \to B_p(R \cdot r)$ is trivial on π_1 and π_2.

Now we discuss the case where M is not orientable. Consider

$$B_p(r) \xrightarrow{i_1} B_p(R \cdot r) \xrightarrow{i_2} B_p(R^2 \cdot r).$$

We can assume at lest one of the three sets are nonorientable. Otherwise we are in a situation we just dealt with. Furthermore, if $B_p(r)$ is orientable, we can consider the following inclusions,

$$B_p(r/R^2) \xrightarrow{j_1} B_p(r/R) \xrightarrow{j_2} B_p(r).$$

We are then in the orientable case. If this happens, we can just choose r smaller. This will not affect our result (we will take this into consideration when choosing R, r). Now we assume that $B_p(r)$ is nonorientable. Therefore all three sets involved are nonorientable.

We consider the first inclusion i_1. We will show $\pi_2(B_p(r)) = 0$. Let us point out here that for this to be true we need the nonorientability, since there are strong topological restrictions on nonorientable three-manifolds. We again prove this by contradiction, along the same line as in the orientable case. If $\pi_2(B_p(r))$ is not trivial, then by the projective plane theorem (which is the nonorientable version of the sphere theorem, Theorem 4.12 in [10]), there is an embedded RP^2 in $B_p(r)$. Again, we need to consider three cases; each will lead to a contradiction.

Case 1. RP^2 does not separate $B_p(R \cdot r)$. We consider the double covers of $B_p(r)$ and $B_p(R \cdot r)$ with the pulled back metric, denoted by $\widetilde{B_p(r)}$ and $\widetilde{B}_p(R \cdot r)$ respectively. Since the double cover of a nonorientable manifold can be constructed as the unit sphere bundle of the determinant bundle, there is a natural lift $\tilde{i}_1$ of i_1, so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{B_p(r)} & \xrightarrow{\tilde{i}_1} & \widetilde{B}_p(R \cdot r) \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ B_p(r) & \xrightarrow{i_1} & B_p(R \cdot r) \end{array}$$

Here $\tilde{i}_1$ is again an inclusion. Now $\widetilde{B_p(r)}$ and $\tilde{B}_p(R \cdot r)$ are subsets in the double cover of $\widetilde{M}$ which is orientable. Note that $\widetilde{B_p(r)} \subset B_{\tilde{p}}^{\widetilde{M}}(4r)$, $\widetilde{B}_p(R \cdot r) \supset B_{\tilde{p}}^{\widetilde{M}}(R \cdot r)$ and the previous argument showed that $B_{\tilde{p}}^{\widetilde{M}}(4r) \to B_{\tilde{p}}^{\widetilde{M}}(R \cdot r)$ induces trivial maps on π_1 and π_2. Thus $\tilde{i}_1$ induces trivial maps on π_1 and π_2. Let $\pi_1^{-1}(RP^2) = S^2$. If this S^2 does not separate $\widetilde{B}_p(R \cdot r)$, then there is a closed curve in $\widetilde{B}_p(R \cdot r)$ intersecting S^2 at only one point. This implies by the Poincaré duality that S^2 is a nontrivial element in $\pi_2(\widetilde{B}_p(R \cdot r))$. This contradicts the fact that $\tilde{i}_1$ induces a trivial map on π_2. If the S^2 separates $\widetilde{B}_p(R \cdot r)$, then both the two components necessarily have nontrivial intersections with $\partial \widetilde{B}_p(R \cdot r)$. In fact, if one component is compact with S^2 as its boundary (namely, does not intersect $\partial \widetilde{B}_p(R \cdot r)$), then projecting it down, we get $B_p(R \cdot r)$ as the union of a compact set and a noncompact set having RP^2 as the common boundary. This means that RP^2 separates $B_p(R \cdot r)$. This contradicts the assumption. Thus S^2 separates $B_p(\widetilde{R} \cdot r)$ into two components both having nontrivial intersection with $\partial \widetilde{B}_p(R \cdot r)$. This is impossible by Lemma 3.2.

Case 2. RP^2 separates $B_p(R \cdot r)$ into two components both having nontrivial intersection with $\partial B_p(R \cdot r)$. This is impossible by Lemma 3.2.

Case 3. RP^2 separates $B_p(R \cdot r)$ into two components: one of them has nonempty intersection with $\partial B_p(R \cdot r)$, and the other, denoted by V, is compact with boundary RP^2. We consider two cases separately.

The first case is when $\pi_1(V)$ is finite. Since V is nonorientable, it follows from the topology of three-manifolds that ∂V consists of two RP^2's [10, p. 77(i)]. This contradicts that $\partial V = RP^2$.

The second case is when $\pi_1(V)$ is infinite. As before, we have the following commuting diagram:

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\tilde{i}} & \widetilde{B}_p(R\cdot r) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ V & \xrightarrow{i_1} & B_p(R\cdot r) \end{array}$$

Then $\partial\widetilde{V} = S^2$ and this S^2 separates $\widetilde{B}_p(R\cdot r)$ into a compact $\widetilde{V}$ and a manifold K which has nontrivial intersection with $\partial\widetilde{B}_p(R\cdot r)$. Since $\widetilde{B}_p(R\cdot r) = \widetilde{V}\#K$, we have $\widetilde{B}_p(R^2\cdot r) = \widetilde{V}\#K_1$ for some K_1. Thus, the inclusion $\pi_1(\widetilde{V}) \to \pi_1(B_{\tilde{p}}^{\widetilde{M}}(R^2\cdot r))$ is injective. Note that $\widetilde{V} \subset B_{\tilde{p}}^{\widetilde{M}}(2Rr)$ and $\pi_1(\widetilde{V})$ is infinite by assumption. This is impossible by Lemma 3.3.

We thus have proved that if $\pi_2(B_p(r)) \neq 0$, then we will get a contradiction in all three cases. Hence $\pi_2(B_p(r)) = 0$, and $B_p(r)$ is a $K(\pi, 1)$ space. It follows that $\pi_1(B_p(r))$ is torsion free (see the argument in the proof of Theorem 2). The same argument shows that $\pi_1(B_p(R\cdot r))$ is torsion free, so that $\pi_1(B_p(r)) \xrightarrow{(i_1)_*} \pi_1(B_p(R\cdot r))$ is a trivial map since otherwise it would contradict Lemma 3.3.

Let us summarize the nonorientable case. We have proved that if

$$R \geq 4\max\{3R_1, R_2, 4N\}, \qquad r \leq \min\left\{\frac{r-1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\right\},$$

then either $B_p(r) \to B_p(R^2\cdot r)$ is trivial on π_1 and π_2 (this happens when both balls are orientable or both are nonorientable), or $B_p(r/R^2) \to B_p(r)$ is trivial on π_1 and π_2. Thus the composition of the two inclusions,

$$B_p\left(\frac{r}{R^2}\right) \to B_p(r) \to B_p(R^2\cdot r),$$

always induces trivial maps on π_1 and π_2.

Thus, we have proved that, no matter whether M is orientable or not, $B_p(r) \xrightarrow{I} B_p(R\cdot r)$ induces trivial maps on π_1 and π_2 when

$$R \geq (4\max\{3R_1, R_2, 4N\})^4, \qquad r \leq r_0 \leq \min\left\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\right\}.$$

We now show that for $r \leq r_0/R$, $B_p(r)$ is contractible in $B_p(R^2\cdot r)$. In fact, from the condition on r, i_1 and i_2 the two inclusions,

$$B_p(r) \xrightarrow{i_1} B_p(R\cdot r) \xrightarrow{i_2} B_p(R^2\cdot r),$$

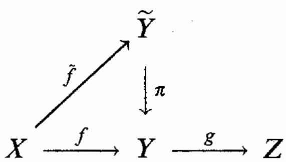
both induce trivial maps on π_1 and π_2. Take a smoothing ρ_* of the distance function ρ, and consider a regular value c of ρ_* such that $\rho_*^{-1}([0, c]) \supset B_p(r)$. Then $\rho_*^{-1}([0, c])$ is a *smooth* three-manifold with nonempty boundary. A well-known theorem in Morse theory (Theorem 23.5 in [11]) implies that $\rho_*^{-1}([0, c])$ has the homotopy type of a two-dimensional CW complex. Thus $B_p(r)$ also has the homotopy type of a two-dimensional CW complex. The same is true for $B_p(R \cdot r)$ and $B_p(R^2 \cdot r)$. Proposition 3.1 (with $R = [4\max\{3R_1, R_2, 4N\}]^8$, $r_0 = \min\{r_1/R^2, r_2/R^3, d/2R^4\}$) is an immediate consequence of the following lemma.

Lemma 3.6. *Let X, Y, Z be two-dimensional CW complexes, and f, g continuous maps,*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

such that f induces a trivial map on π_1 and g induces a trivial map on π_2. Then $g \circ f$ is homotopic to a constant map.

Proof. Since f is trivial on π_1, we have the lifting $\tilde{f}$ such that the following diagram commutes:



where $\widetilde{Y}$ is the universal covering space of Y. It thus suffices to prove that $g \circ \pi \circ \tilde{f}$ is homotopic to a constant map. Denote $\psi = g \circ \pi : \widetilde{Y} \to Z$. Then

$$\psi_*(\pi_2(\widetilde{Y})) = g_* \circ \pi_*(\pi_2(\widetilde{Y})) = g_*(\pi_2(Y)) = e;$$

that is, ψ is trivial on π_2. We now show that ψ is homotopic to a constant map. Since $\widetilde{Y}$ is a two-dimensional CW complex which is simply connected, by Corollary 3.6 on p. 221 of [21], $\widetilde{Y}$ is homotopy equivalent to the wedge of S^2's, $\widetilde{Y} = S^2 \vee \cdots \vee S^2$. Each of these S^2 represents an element in $\pi_2(\widetilde{Y})$. Since ψ is trivial on $\pi_2(\widetilde{Y})$, it follows that ψ, when restricted on each S^2, is homotopic to a constant map. Therefore ψ is homotopic to a constant map. Hence $g \circ f$ is homotopic to a constant map.

Proof of Theorem 1. The argument from Proposition 3.1 to Theorem 1 is somewhat formal. It is essentially the same for all types of finiteness theorems, namely, a center of mass argument. The observation is that $\mathscr{M}(n)$ is precompact with respect to the Hausdorff distance. And for two manifolds which are Hausdorff close, and geometrically contractible in the sense of Proposition 3.1, we can construct a map between them which is a homotopy equivalence. This can be easily seen from the point of view of obstruction theory. Proposition 3.1 guarantees that there is no obstruction for extending maps. We can thus start constructing the map skeleton-wise. The detail is carried out by P. Petersen in [13]. This completes our proof of Theorem 1.

Remark. Proposition 3.1 is actually more than what we need to conclude Theorem 1. In fact, the statement that I is trivial on π_1 and π_2 is enough to imply Theorem 1. For this see P. Petersen [13].

Remark. As kindly pointed out by P. Petersen, our argument actually shows there are only finitely many *simple* homotopy types for the class in Theorem 1.

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